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an application to NP optimization problems

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Abstract

We introduce the notion bounded relation which comprises most resource bounded reducibilities which can be found in the literature, including non-uniform bounded reducibilities such as $\leqslant_T^{\mathscr{P}/poly}$. We state conditions on bounded relations which are again satisfied for most bounded reducibilities and which imply that every countable partial ordering can be embedded into every proper interval of the recursive degrees. As corollaries, we obtain that every countable partial ordering can be embedded into every proper interval of (REC, $\leqslant_T^{\mathscr{P}/poly}$), as well as into every proper interval between either two maximization or two minimization problems in the structures $(\mathscr{NPO}, \leqslant_E)$ and $(\mathscr{NPO}, \leqslant_L)$. For the two latter structures, we show further that the result about embeddings of partial orderings extends to embeddings of arbitrary countable distributive lattices where in addition the least or the greatest element of the lattice can be preserved. Among other corollaries, we obtain that for both structures every non-trivial $\mathscr{NP}$ optimization problem bounds a minimal pair. In connection with embeddings into $\mathscr{NPO}$ we introduce a representation of maximization or minimization problems by polynomial time computable functions. We consider decidability issues w.r.t. this representation and show for example that there is no effective procedure which decides for a (subrecursive) index of a polynomial time computable function whether the corresponding maximization problem is approximable within a constant factor.

Keywords: Resource bounded reducibilities; Partial order embeddings; Lattice embeddings; Abstract reducibilities; Non-uniform reducibilities; Approximation preserving reducibilities

1. Introduction

1.1. Overview

A functional Γ is a function which maps every pair of a subset B of the natural numbers ω and a natural number x to a value $\Gamma(B,x)$ in $\{0,1\}$. A functional Γ is

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recursive iff there is an oracle Turing machine which for all inputs x and all oracles B eventually halts and outputs $\Gamma(B,x)$. Following Book et al. [3], we call a binary relation $\leqslant_r$ on 2^ω a BOUNDED REDUCIBILITY iff there is some effectively given list of recursive functionals $\Delta_0, \Delta_1, \ldots$ such that we have for all sets A and B

$$A \leqslant_r B \quad \text{iff} \quad \exists i \in \omega \forall x \in \omega [A(x) = \Delta_i(B,x)], \tag{1}$$

that is, for a bounded reducibility a fact $A \leqslant_r B$ holds iff it is witnessed by some functional in the given list. The concept bounded reducibility on ω^ω is defined in the same way. Most of the usual resource bounded reducibilities such as polynomial time bounded Turing reducibility $\leqslant_T^{\mathscr{P}}$ or logarithmic space bounded many-one reducibility $\leqslant_m^{log}$ are indeed bounded reducibilities.

For a bounded reducibility $\leqslant_r$ as in (1), the lower cone of each set B consists exactly of the sets $\Delta_0(B), \Delta_1(B), \ldots$ and hence is countable. As a consequence reducibilities with uncountable lower cones such as $\leqslant_T^{\mathscr{P}/poly}$ are not bounded reducibilities. In order to comprise relations of the latter type, we extend the concept bounded reducibilities to bounded relations. Then we exhibit a small set of rather general and intuitively meaningful conditions on bounded relations which imply that every countable partial ordering can be embedded into every proper interval of the structure induced on the recursive sets by the bounded relation under consideration.

The concept bounded relation and the result on embeddings of partial orderings extend canonically from binary relations on 2^ω to binary relations on ω^ω. In terms of bounded relations on ω^ω we are able to analyze structural properties of reducibilities between optimization problems such as $\leqslant_L$ and $\leqslant_E$ introduced in [14, 7], respectively. More precisely, we represent maximization problems by functions in ω^ω via a mapping Ψ which takes every element in ω^ω to a maximization problem, and similarly for minimization problems. By choice of the mapping Ψ the relations $\leqslant_L$ and $\leqslant_E$ both induce bounded relations on ω^ω, which we denote by $\leqslant_L^\Psi$ and $\leqslant_E^\Psi$, respectively. Here for example $\leqslant_L^\Psi$ is defined by

$$f \leqslant_L^\Psi g \quad \text{iff} \quad \Psi(f) \leqslant_L \Psi(g). \tag{2}$$

All bounded relations mentioned so far satisfy the assumption of the result on partial order embeddings stated above, and in particular the result holds for the relations $\leqslant_L^\Psi$ and $\leqslant_E^\Psi$. As corollaries, we obtain for the two structures induced on $\mathscr{NPO}$ by the relations $\leqslant_L$ and $\leqslant_E$ that every countable partial ordering can be embedded into every proper interval between either two maximization or two minimization problems. Further, we show that both corollaries can, in fact, be strengthened to embeddings of arbitrary countable distributive lattices. In particular, in case $\mathscr{NPO}$ contains optimization problems which are not reducible to all other optimization problems in $\mathscr{NPO}$, then every such problem bounds a minimal pair of optimization problems in $\mathscr{NPO}$.

1.2. Related work

Mehlhorn [11] states axioms for bounded reducibilities and shows by a Ladner-style construction that his axioms imply density of the recursive degrees, that is, every proper interval of the recursive degrees contains an intermediate degree. He states further that his axioms, in fact, imply the embeddability of arbitrary countable partial orderings into every proper interval of the recursive degrees. Our result on partial order embeddings extends Mehlhorn's corresponding result in so far as his axioms are designed to be applied to bounded reducibilities of Turing type and for example are neither satisfied for bounded reducibilities of many-one type, nor for bounded relations such as $\leqslant_T^{\mathscr{P}/poly}$. Note, however, that the formulation of our abstract approach relies strongly on Mehlhorn's concept of delayed simulation.

Generalizing previous results due to Ladner [9], Landweber et al. [10], and others, Ambo-Spies [1] shows for various polynomial time bounded reducibilities that every countable distributive lattice can be embedded (in fact, with least or greatest element preserved) into every proper interval of the structure induced on the recursive sets. Consequently, in particular, all countable partial orderings can be embedded into such intervals and one obtains as corollaries the restrictions of Mehlhorn's result on partial order embeddings to the polynomial time bounded reducibilities under consideration. It is shown in [12, 13] that the lattice embedding result due to Ambo–Spies extends to the abstract setting. Here in the case of bounded reducibilities the assumptions used are rather natural, whereas in the case of bounded relations considered in [12] the formulation of the assumptions becomes more technical. Regarding the small number of intended applications, in the following we refrain from stating the results on lattice embeddings for bounded relations in full generality and consider lattice embeddings only in the context of the specific bounded relations $\leqslant_L^{\Psi}$ and $\leqslant_E^{\Psi}$.

Like in [1], we show our embedding results via embedding the countable atomless Boolean algebra by means of the gap language technique, which provides a more modular approach to looking-back arguments as introduced by Ladner [9]. Constructions of this type yield intermediate sets which are rather artificial mixtures of sets bounding the given interval, and a similar remark holds for intermediate optimization problems constructed this way. In contrast to this, Crescenzi et al. [4] introduce the reducibility $\leqslant_{AP}$ and show from the assumption that the polynomial time hierarchy does not collapse that there are natural problems such as Minimum Bin Packing and Minimum Edge Coloring which are neither $\leqslant_{AP}$-complete for the class $\mathscr{APX}$, nor are optimally solvable in polynomial time.

Our methods extend neither to the reducibility $\leqslant_{AP}$, to *P*-reducibility as introduced in [5], nor to *PTAS*-reducibility as introduced in [6]. The definitions of these reducibilities follow the usual scheme for reducibilities between optimization problems: instances are mapped to instances and then solutions are mapped back to solutions such that the quality of the solutions is related in some prescribed way. The reducibilities mentioned, however, are defined in terms of function classes which cannot be effectively listed. For example in the cases of *P*- and *PTAS*-reducibility, the quality of the solutions is required to be related by some arbitrary and by some recursive function, respectively. As a consequence we are not able to analyze these reducibilities in terms of bounded relations in the same way as for the reducibilities $\leqslant_L$ and $\leqslant_E$. Note in this connection that in [5] a reducibility between optimization problems is introduced where the quality of the solutions is related by an arbitrary function from the set of rationals in the open interval between 0 and 1 to itself. Then, under the assumption that $\mathscr{P}$ differs from $\mathscr{NP}$, the existence of intermediate problems in $\mathscr{NPO}$ w.r.t. this reducibility is shown by a Ladner-style looking-back construction. While this result might be correct, the proof as it is stated seems to require an enumeration of all functions from the set of rationals in the open interval between 0 and 1 to itself.

1.3. Notation

We denote the set of natural numbers and its powerset by ω and 2^ω, respectively. We identify natural numbers with binary strings in $\{\lambda, 0, 1, 00\ldots\}$ by means of the order isomorphism which takes the usual ordering on the natural numbers to the length-lexicographical ordering. Functions are always meant to be total, unless explicitly attributed as being partial. By ω^ω we refer to the class of functions from ω to ω. We denote subsets of ω as sets, and subsets of 2^ω or ω^ω as classes, for short. We identify sets with their characteristic function, and accordingly we view 2^ω as a subset of ω^ω and apply concepts which are introduced for functions also to sets. For functions f and g in ω^ω, we write $f =^* g$ if f and g agree on all but finitely many places.

For numbers $x_1, \ldots, x_n$ in ω we denote by $\langle x_1, \ldots, x_n \rangle$ the code obtained by applying the standard effective and effectively invertible bijection from ω^n to ω. The join of functions f_0 and f_1 in ω^ω is defined by

$$\oplus(f_0, f_1)(x) := \begin{cases} 0 & \text{if } x = \lambda, \\ f_0(y) & \text{if } x = 0y, \\ f_1(y) & \text{if } x = 1y. \end{cases}$$

By lower-case Greek letters $\alpha, \beta, \gamma, \ldots$ we denote PARTIAL FUNCTIONS (from ω to ω), that is, functions from some subset I of ω to ω. We denote the DOMAIN of a partial function α by $dom(\alpha)$. A partial function is FINITE iff its domain is finite. A partial function is a PARTIAL CHARACTERISTIC FUNCTION if its range is contained in the set $\{0, 1\}$. For partial functions α, β and a set M, we let

$$\langle \alpha, \beta \rangle^M(x) := \begin{cases} \alpha(x) & \text{if } x \text{ is in } M, \\ \beta(x) & \text{otherwise,} \end{cases} \tag{3}$$

that is, $\langle \alpha, \beta \rangle^M$ is the partial function which agrees with α on $dom(\alpha) \cap M$, with β on $dom(\beta) \cap \overline{M}$, and is undefined otherwise. Given a function g in ω^ω and a partial function β, we refer to the function

$$\langle g, \beta \rangle := \langle g, \beta \rangle^{\omega \setminus dom(\beta)} = \langle \beta, g \rangle^{dom(\beta)}$$

as $\boldsymbol{\beta}$-PATCH of g, that is, $\langle g, \beta\rangle$ is the unique function which agrees with β for all arguments in $dom(\beta)$ and which agrees with g, otherwise.

2. Partial order embeddings for bounded relations

2.1. Bounded relations

We introduce concepts which will be used in the formulation and proof of our result on partial order embeddings for bounded relations. In connection with Definition 1, note that $\otimes$ denotes the Cartesian product. Further recall that a functional on ω^ω is a function from $\omega^\omega \otimes \omega$ to ω and that such a functional Γ is recursive iff there is an oracle Turing machine which for all functions f and all natural numbers x, computes the value $\Gamma(f,x)$ with function oracle f and input x.

Definition 1.

- A predicate R on $(\omega^\omega)^2 \otimes \omega^2$ is recursive iff there is some recursive functional Γ where we have for all f and g in ω^ω and for all x and y in ω

$$R(f,g,x,y) \quad \text{iff} \quad \Gamma(f \oplus g, \langle x,y\rangle) = 1.$$

- A binary relation $\leqslant_r$ on ω^ω is a BOUNDED RELATION (ON $\boldsymbol{\omega^\omega}$) if there is some recursive predicate R on $(\omega^\omega)^2 \otimes \omega^2$ such that we have for all functions f and g in ω^ω

$$f \leqslant_r g \quad \text{iff} \quad \exists j \in \omega \ \forall k \in \omega R(f,g,j,k).$$

- The concept BOUNDED RELATION ON $\mathbf{2^\omega}$ is defined in like manner.

The following examples indicate that indeed most of the resource bounded reducibilities which can be found in the literature are bounded relations.

- Bounded reducibilities are by definition bounded relations because for an effectively given list of recursive functionals $\Delta_0, \Delta_1, \ldots$, the matrix of the right-hand side of (1) is a recursive predicate. Thus in particular all the usual resource bounded reducibilities such as $\leqslant_T^{\mathscr{P}}$ or $\leqslant_m^{log}$ are bounded relations on 2^ω.
- The non-uniform reducibility $\leqslant_T^{\mathscr{P}/poly}$ is a bounded relation on 2^ω as is witnessed by the recursive predicate R where $R(A,B,\langle e,c\rangle,n)$ holds iff the eth polynomial time bounded oracle Turing machine computes the restriction of A to strings of length n with the help of some advice string z_n of length less than n^c.

We show in Section 3 that also the relation $\leqslant_L^\Psi$ defined in (2) is a bounded relation on ω^ω, and that accordingly our results on embeddings of partial orderings for bounded relations carry over to the relation $\leqslant_L$. A similar remark holds for the relations $\leqslant_E^\Psi$ and $\leqslant_E$.

2.2. Gap languages and diagonalization

Next, we introduce *gap languages* and some related notation. The use of gap languages is a standard technique for showing embedding results and has been employed in connection with several specific bounded reducibilities; see the section on uniform diagonalization in [2], as well as Schöning [16, 17].

Definition 2.
- A GAP LANGUAGE is a subset of ω which is infinite and co-infinite.
- Let A be some set. A block of A is a maximal set of consecutive natural numbers which either all are in A or all are in the complement of A.
- Let A and B be gap languages. The set B is a GAP COVER for A iff every block of B contains some block of A.

We number the blocks of a set A in the natural way, starting with block 0, and thus for example the number 0 is always contained in block 0. Obviously, a set has infinitely many blocks iff it is a gap language.

Definition 3.
- Let G be a gap language and let g, g', h, and h' be functions in ω^ω. The pairs (g,h), and (g',h') are G-similar, written $(g,h) \simeq^G (g',h')$, iff there are infinitely many blocks of G where f agrees with f' and g agrees with g'.
- A subclass $\mathscr{C}$ of ω^ω is EFFECTIVELY COMPACT if $\mathscr{C}$ is equal to $\bigotimes_{i\in\omega} C_i$ where the sets $C_0, C_1, \ldots$ are non-empty and finite and where given an index i we can compute a list of the elements in C_i.
- A subclass $\mathscr{C}$ of ω^ω is closed under finite variations (C.F.V.) iff for all functions f and f' in ω^ω, the facts $f =^* f'$ and f in $\mathscr{C}$ together imply f' in $\mathscr{C}$. A binary relation $\leqslant_r$ on ω^ω is c.f.v. iff for all functions f, f', g, and g' in ω^ω, the facts $f =^* f'$, $g =^* g'$, and $f \leqslant_r g$ together imply $f' \leqslant_r g'$. For subclasses of 2^ω and for binary relations on 2^ω, the concept closure under finite variations is defined accordingly.

Lemma 4 (Diagonalization lemma). *Let $\leqslant_r$ be a bounded relation which is c.f.v., and let f and g be recursive functions where $g \not\leqslant_r f$. Let $\mathscr{C}$ be an effectively compact subclass of ω^ω. Then there is a recursive gap language G such that for all functions f' and g' in $\mathscr{C}$ we have*

$$(f,g) \simeq^G (f',g') \quad \textit{implies} \quad g' \not\leqslant_r f'.$$

In the proofs of subsequent embedding results, we will use the diagonalization lemma in order to show that the constructed embedding preserves non-order.

Proof of Lemma 4. We choose some recursive relation R which witnesses that $\leqslant_r$ is a bounded relation. We construct in stages a gap language G as required in the lemma.

During stage s, we specify which numbers are in block s of G. This then determines G by letting $G(0)=0$. We denote block s of G by I_s, and at stage 0, we let I_0 be equal to $\{0\}$. At stage $s>0$, by effective compactness of C, we compute the finitely many partial functions where, firstly, their domain is equal to the union of the sets $I_0,\dots,I_{s-1}$ and, secondly, which are restrictions of functions in $\mathscr{C}$. For all pairs (α,β) of such partial functions and for all $j<s$, we let $n_{\alpha,\beta,j}$ be the least number such that

$$R(\langle g,\beta\rangle,\langle f,\alpha\rangle,j,n_{\alpha,\beta,j}) \tag{4}$$

is false. There is always such a number $n_{\alpha,\beta,j}$, because otherwise j witnesses that $\langle g,\beta\rangle$ is reducible to $\langle f,\alpha\rangle$, which by $\leqslant_r$ being c.f.v. in turn implies $g\leqslant_r f$, thus contradicting our assumption on g and f. Furthermore, the least such number can be found effectively in α, β, and j, because R is recursive. Next, we choose the finite block I_s large enough such that for all $j<s$ and all pairs (α,β) as above the value of (4) is determined by the restriction of the corresponding function arguments to the union of $I_0,\dots,I_s$.

In order to show that the gap language G has the required properties, assume for a proof by contradiction that there are functions f' and g' in $\mathscr{C}$ where, firstly, (f',g') and (f,g) are G-similar and, secondly, the number j witnesses that g' is $\leqslant_r$-reducible to f'. Choose some $s\geq j$ where f' and g' agree with f and g, respectively, on block s of G. Let α and β be the restrictions of f' and g', respectively, to the union of the blocks $I_0,\dots,I_{s-1}$. Now, the witness $n_{\alpha,\beta,j}$ we found during stage s of the construction of G witnesses $g'\not\leqslant_r f'$, because $\langle f,\alpha\rangle$ and $\langle g,\beta\rangle$ agree with f' and g', respectively, on the relevant blocks $I_0,\dots,I_s$ of G. $\square$

2.3. The class of admissible cases and delayed simulations

Recall from the introduction that $\langle f,g\rangle^M$ is the function which agrees with f for all places in M, and agrees with g for all places in the complement of M.

Definition 5. Let $\leqslant_r$ be a binary relation on ω^ω.

- The class of ADMISSIBLE CASES of $\leqslant_r$ is

$$\mathscr{M}_r := \{M\subseteq\omega\text{: for all } f,g,h \text{ in } \omega^\omega, f\leqslant_r h \text{ and } g\leqslant_r h \\ \text{together imply } \langle f,g\rangle^M\leqslant_r h\}.$$

- The CLASS OF LEAST FUNCTIONS of $\leqslant_r$ is

$$\mathscr{L}_r := \{f \text{ in } \omega^\omega\text{: } f\leqslant_r g \text{ for all } g \text{ in } \omega^\omega\}.$$

We extend the concepts introduced in Definition 5 to bounded relations on 2^ω. Here we universally quantify over all sets in 2^ω instead of over functions in ω^ω.

Proposition 6. *Let $\leqslant_r$ be a binary relation on 2^ω or on ω^ω.*
- *If $\mathscr{L}_r$ contains (the characteristic functions of) $\emptyset$ and ω, then $\mathscr{M}_r$ is a subclass of $\mathscr{L}_r$.*
- *The class $\mathscr{M}_r$ contains $\emptyset$ and is closed under the set theoretical operations union, intersection, and complement. Equivalently, $(\mathscr{M}_r, \subseteq)$ is a subalgebra of $(2^\omega, \subseteq)$.*

Proof. We give the proof for relations on ω^ω and omit the almost identical proof for relations on 2^ω. Concerning the first assertion, by assumption we have for all functions f and for all sets M in $\mathscr{M}_r$

$$M = \langle \omega, \emptyset \rangle^M \leqslant_r f.$$

The second assertion follows from the definition of $\mathscr{M}_r$ and because we have for all functions f, g, M, M_0, and M_1

$$\langle f, g \rangle^{\emptyset} = g, \quad \langle f, g \rangle^{\overline{M}} = \langle g, f \rangle^M, \quad \langle f, g \rangle^{M_0 \cap M_1} = \langle \langle f, g \rangle^{M_0}, g \rangle^{M_1},$$

where the equations show, from left to right, that $\mathscr{M}_r$ contains $\emptyset$ and is closed under complementation and intersection; closure under union then follows by the De Morgan formula. $\square$

Observe in connection with the first assertion in Proposition 6 that for specific resource bounded reducibilities we often find that $\mathscr{M}_r$ coincides with $\mathscr{L}_r$, and for example $\mathscr{M}_T^{\mathscr{P}}$ is equal to $\mathscr{P}$. However, there are counterexamples which show that this statement is false for bounded reducibilities in general, see [12] or [13].

The concept delayed simulation introduced in Definition 7 is due to Mehlhorn [11]. Delayed simulations can be viewed as an abstract version of the ability to compute an arbitrary recursive set, however delayed, within rather restrictive time or space bounds.

Definition 7.
- Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be the standard enumeration of the partial recursive functions from ω to $\{0, 1\}$.
- A set S is a delayed simulation of some set A iff there is some non-decreasing function l with range ω such that for all x we have $S(x) = A(l(x))$.
- A subclass $\mathscr{M}$ of 2^ω is a SIMULATION CLASS iff there is some recursive function *sim* such that
  1. $\varphi_{sim(e)}$ is in $\mathscr{M}$ for all e in ω,
  2. if φ_e is a set where $\varphi_e(0) = 0$, then $\varphi_{sim(e)}$ is a delayed simulation of φ_e.

Note that, for example, the class of sets computable in polynomial time and the class of sets computable in logarithmic space are both simulation classes, for proofs and further discussion see [12] or [13].

Lemma 8 (Coding lemma). *Let $A_0, A_1, \ldots$ be a uniformly recursive sequence of sets and let G be a gap language. Let $\mathscr{M}$ be a simulation class which contains all finite*

sets and where the structure $(\mathscr{M}, \subseteq)$ *is a subalgebra of* $(2^\omega, \subseteq)$. *Then there are sets* $R_0, R_1, \ldots$ *in* $\mathscr{M}$ *and a gap language* M *in* $\mathscr{M}$ *such that*
- *the set* M *is a gap cover for* G,
- *for all* i *and* s *in* ω *and for all* x *in block* s *of* M, *we have* $R_i(x) = A_i(s)$.

The point of the coding lemma is that it yields delayed simulations R_i of the sets A_i which are "synchronized" via the gap language M, that is, for all i and s the set R_i is constant on block s of M and has the value $A_i(s)$ there. The sets R_i in the coding lemma are uniformly recursive, because the sets A_i are, and due to the second condition in the conclusion of the coding lemma.

Due to space considerations, we omit the lengthy proof of the coding lemma. The proof, which involves an application of the recursion theorem, can be found in [12, 13]. Observe that in the case of the simulation class $\mathscr{P}$ of sets computable in polynomial time the statement of the coding lemma can be shown as follows: first, we construct a recursive gap cover L for G where the blocks of L are so large that we can compute each of the values $A_0(s), \ldots, A_s(s)$ in time equal to the length of the least element of block s of L; next, we obtain a gap cover M for L in $\mathscr{P}$ by a standard looking-back construction. Then by construction the sets R_i which agree with $A_i(s)$ on block s of M are all computable in polynomial time.

2.4. Embeddings of partial orderings

We will show next that for a wide class of bounded relations every countable partial ordering can be embedded into every proper interval of the recursive degrees. Before, we introduce some notation.

Definition 9. The join operation $\oplus$ is a L.U.B.-OPERATION for a binary relation $\leqslant_r$ on ω^ω iff the join of two functions is always a least upper bound for them, that is,
- for all f and g, we have $f \leqslant_r f \oplus g$ and $g \leqslant_r f \oplus g$,
- for all f, g, and h, the facts $f \leqslant_r h$ and $g \leqslant_r h$ together imply $f \oplus g \leqslant_r h$.

Definition 10. Let $\leqslant_r$ be a partial preordering on ω^ω, that is, let $\leqslant_r$ be reflexive and transitive.
- The partial preordering $\leqslant_r$ is FAITHFUL iff $\leqslant_r$ has the join operation as a l.u.b.-operation and $\mathscr{L}_r$ contains all constant functions.
- We denote

  $$deg_r(f) := \{h \in \omega^\omega\colon\ f \leqslant_r h \text{ and } h \leqslant_r f\}$$

  as $\leqslant_r$-DEGREE of the function f. The relation $\leqslant_r$ induces canonically a partial ordering on degrees, which we denote by $\leqslant$. A degree is recursive iff it contains a recursive function.

All concepts introduced in Definition 10 extend in the natural way to relations on 2^ω. The term faithful refers to the fact that for faithful relations to some extent "easy" functions are reducible to more complex ones. Faithful partial preorderings are a special case of faithful relations as considered in [12, 13], however, the formulation used to define the concept of faithfulness is slightly more involved in the general case of a not necessarily transitive relation. While it is shown there that results on partial ordering and lattice embeddings extend to non-transitive relations by basically the same proofs as in the transitive case, in the non-transitive case proofs tend to become more technical. So we restrict the exposition here to transitive relations, which then in particular allows the formulation of results and proofs in terms of the usual concept of degree.

Theorem 11. *Let $\leqslant_r$ be a bounded relation on either 2^ω or ω^ω such that $\leqslant_r$ is a faithful partial preordering which is c.f.v. and where $\mathscr{M}_r$ is a simulation class. Then every countable partial ordering can be embedded into every proper interval of the recursive $\leqslant_r$-degrees.*

Proof. We show Theorem 11 for the case of a bounded relation on ω^ω and omit the almost identical proof for bounded relations on 2^ω. We write $A \subseteq^* B$ iff all but finitely many numbers which are in A are also in B. By $[A]$, we denote the class of all sets which are finite variations of A, that is, the equivalence class of A w.r.t. the relation $=^*$, and by $\leqslant^*$ we denote the partial ordering on equivalence classes which is induced by the relation $\subseteq^*$. We let $\mathscr{P}^*$ be equal to $\{[A]\colon A \text{ in } \mathscr{P}\}$ where $\mathscr{P}$ is the class of polynomial time computable sets. In [1], it is shown that the structure $(\mathscr{P}^*, \leqslant^*)$ is the countable atomless Boolean algebra. More precisely, $(\mathscr{P}^*, \leqslant^*)$ inherits the property of being a Boolean algebra from the structure $(\mathscr{P}, \subseteq)$, and it does not contain atoms, because given some infinite recursive set A, by a standard looking-back construction, we can construct a subset B of A in $\mathscr{P}$ such that the sets B and $A \backslash B$ are both infinite, that is, B is strictly above the empty set and is strictly below A w.r.t. the relation $\subseteq^*$. It is known from lattice theory that every countable partial ordering can be embedded as a partial ordering into the countable atomless Boolean algebra, see [1] and the references given there. Now embeddings of partial orderings compose, and thus we are done if we can embed the structure $(\mathscr{P}^*, \leqslant^*)$ in the required way.

We choose recursive functions f and g from the degrees which bound the given interval of the degree structure such that f is $\leqslant_r$-reducible to g, but not vice versa. By $\leqslant_r$ being transitive faithful, the degree of g is equal to the degree of $f \oplus g$, and the same holds for f and $f \oplus \emptyset$, where as usual we identify $\emptyset$ with the constant function with value 0. So we obtain a gap language G by applying the diagonalization lemma to the functions $f \oplus \emptyset$ and $f \oplus g$ and to the effectively compact class

$$\mathscr{C} := \bigotimes_{x \in \omega} \{(f \oplus \emptyset)(x), (f \oplus g)(x)\}.$$

Next, we apply the coding lemma to the gap language G and to an appropriate effective listing $A_0, A_1, \ldots$ of $\mathscr{P}$ and obtain a gap language M in $\mathscr{M}_r$ and sets R_i in $\mathscr{M}_r$. We define a function from $\mathscr{P}$ to ω^ω by

$$\Pi_0 : A_i \mapsto \langle f \oplus g, f \oplus \emptyset \rangle^{R_i} \tag{5}$$

and a function Π from $\mathscr{P}^*$ to $\leqslant_r$-degrees by

$$\Pi : [A_i] \mapsto deg_r(\Pi_0(A_i)).$$

We show that Π is the embedding of $(\mathscr{P}^*, \leqslant^*)$ we are looking for. For all i and n in ω, the function $\Pi_0(A_i)$ agrees by definition on block n of M with $f \oplus g$ iff n is in A_i, and agrees there with $f \oplus \emptyset$, otherwise.

Claim 1. *The function Π is well-defined.*

Proof. If $[A_i]$ is equal to $[A_j]$, that is, if A_i is a finite variation of A_j, then by the preceding remark the functions $\Pi_0(A_i)$ and $\Pi_0(A_j)$ disagree at most at finitely many places, and thus are in the same $\leqslant_r$-degree by $\leqslant_r$ being c.f.v.

It remains to show that Π is a function into the given interval, and that Π respects order and non-order. By definition of Π it is sufficient to show corresponding assertions for Π_0.

Claim 2. *For every function h in the range of Π_0 we have $f \leqslant_r h \leqslant_r g$.*

Proof. Let h be some function in the image of Π_0. By definition of Π_0 and of the join operation, h can be written as $f \oplus z$ for some function z in ω^ω, and hence f is reducible to h due to $\leqslant_r$ being faithful. Further h is reducible to g, because $f \oplus \emptyset$ and $f \oplus g$ are, and because we have chosen the sets R_i in the class $\mathscr{M}_r$.

Claim 3. *The function Π respects order.*

Proof. Given A_i and A_j where $[A_i] \leqslant^* [A_j]$, that is, where $A_i \subseteq^* A_j$, we infer

$$\Pi_0(A_i) := \langle f \oplus g, f \oplus \emptyset \rangle^{R_i} =^* \langle \Pi_0(A_j), f \oplus \emptyset \rangle^{R_i} \leqslant_r \Pi_0(A_j), \tag{6}$$

from which $\Pi_0(A_i) \leqslant_r \Pi_0(A_j)$ follows because $\leqslant_r$ is c.f.v. Concerning (6), by assumption on A_i and A_j and by choice of the sets $R_0, R_1, \ldots$ we have $R_i \subseteq^* R_j$. Further, by definition of Π_0 the function $\Pi_0(A_j)$ agrees with $f \oplus g$ for all places x in R_j, and hence for almost all places x in R_i. The relations in (6) then hold, from left to right, by definition of Π_0, by the preceding remark, and finally because R_i is in $\mathscr{M}_r$ and because by Claim 2 the function f and hence by assumption on $\leqslant_r$ also $f \oplus \emptyset$ are $\leqslant_r$-reducible to $\Pi_0(A_j)$.

Claim 4. *The function Π respects non-order.*

Proof. Given A_i and A_j where $[A_i]\not\leqslant^*[A_j]$, that is, where $A_i\not\subseteq^* A_j$, we infer by the introductory remark that there are infinitely many blocks of M on which $\Pi_0(A_i)$ agrees with $f\oplus g$ and $\Pi_0(A_j)$ agrees with $f\oplus\emptyset$, that is, we have

$$(f\oplus g, f\oplus\emptyset) \simeq^M (\Pi_0(A_i),\Pi_0(A_j)). \tag{7}$$

We have chosen M as a gap cover for the gap language G, that is, each block of M contains some block of G, and consequently, (7) remains valid with M replaced by G. Now G has been obtained by applying the diagonalization lemma to the effectively compact class $\mathscr{C}$ which by construction contains the functions $\Pi_0(A_i)$ and $\Pi_0(A_j)$, and consequently $\Pi_0(A_i)$ is not reducible to $\Pi_0(A_j)$. □

Corollary 12. *Every countable partial ordering can be embedded into every proper interval of the recursive $\leqslant_T^{\mathscr{P}/poly}$-degrees.*

Proof. We show that the relation $\leqslant_T^{\mathscr{P}/poly}$ satisfies the assumption of Theorem 11. By definition, a set A is $\leqslant_T^{\mathscr{P}/poly}$-reducible to a set B if there is some polynomial time bounded oracle Turing machine T and some polynomial p such that for all n there is some advice string z_n of length less or equal to $p(n)$ where for all strings x of length n, T computes $A(x)$ on number input $\langle x,z_n\rangle$ and oracle B. Thus the recursive predicate R witnesses that $\leqslant_T^{\mathscr{P}/poly}$ is a bounded relation where $R(A,B,\langle e,c\rangle,n)$ holds iff the eth polynomial time bounded oracle Turing machine computes the restriction of A to strings of length n while using some advice string z_n of length less than n^c. We leave it to the interested reader to show that $\leqslant_T^{\mathscr{P}/poly}$ is transitive faithful and c.f.v., and that $\mathscr{M}_T^{\mathscr{P}/poly}$ contains the simulation class $\mathscr{P}$ and thus is itself a simulation class. □

3. Approximation preserving reducibilities

3.1. NP optimization problems

We consider reducibilities between OPTIMIZATION PROBLEMS. Here the latter are regarded as four-tuples which consist of:

- a set of binary strings, where strings in the set are denoted as instances of the optimization problem under consideration,
- a relation sol between strings where y is denoted as feasible solution for the instance x iff $\mathrm{sol}(x,y)$ is true,
- a function m which assigns to each pair (x,y) where $\mathrm{sol}(x,y)$ is true a positive integer $m(x,y)$ meant as measure or quality of the solution y w.r.t. instance x,
- a goal which is either to minimize or to maximize the measure of the feasible solutions for any given instance.

An optimization problem is in the class $\mathscr{NPO}$ of NON-DETERMINISTIC POLYNOMIAL TIME OPTIMIZATION PROBLEMS iff the set of instances, the set of pairs (x,y) where y is a feasible solution for the instance x, and the measure m are all computable in polynomial time, and in addition the length of the feasible solutions for any instance x is bounded in the length of x by some fixed polynomial which does not depend on x (see for example [6]).

In the sequel, we call an instance of an optimization problem degenerated iff its set of feasible solutions is empty. For every non-degenerated instance x of some given maximization problem, we let $m^*(x)$ be the value of some optimal solution of x, that is, for example in the case of a maximization problem we have

$$m^*(x) := \max_{\{y \in \omega : \mathrm{sol}(x,y)\}} m(x, y).$$

Papadimitriou and Yannakakis [14] introduce a reducibility $\leqslant_L$ between optimization problems. According to their definition, an optimization problem P is $\leqslant_L$-reducible to an optimization problem Q iff there are polynomial time computable functions r and p and a rational $c > 0$ such that we have for every instance x of P

- $r(x)$ is an instance of Q where $m^*(r(x)) \leqslant c \cdot m^*(x)$,
- for every feasible solution y of $r(x)$, the number $p(x, y)$ is a feasible solution of x where

$$|m^*(x) - m(x, p(x, y))| \leqslant c \cdot |m^*(r(x)) - m(r(x), y)|.$$

Recall from complexity and recursion theory that many-one reducibilities are usually defined by specifying some subclass of ω^ω: a set A is reducible to some set B iff there is some function h in the class considered where for all x, the number x is in A iff $h(x)$ is in B. By this definition, a nonempty set cannot be reducible to the empty set and likewise every set different from ω is not reducible to ω, which is, for example, at variance with the intuitive requirement that finite and co-finite sets should be reducible to all other sets. Thus it is common usage to adjust the definition of the many-one reducibility under consideration such that all "easy" sets are reducible to the sets ω and $\emptyset$. For similar reasons, we will adjust the original definition of the reducibility $\leqslant_L$ in order to ensure a more natural behavior for optimization problems where the set of instances is empty or contains degenerated instances.

Definition 13. An optimization problem P is $\leqslant_L$-reducible to some optimization problem Q iff there are polynomial time computable functions r and p and a rational $c > 0$ where for every instance x of P we have

- in case $r(x)$ is the empty string, then x has an empty set of feasible solutions,
- in case $r(x)$ has the form $0z$, then
  - (i) z is an instance of Q where $m^*(z) \leqslant c \cdot m^*(x)$,
  - (ii) for every feasible solution y of z, the number $p(x, y)$ is a feasible solution of x where we have

$$|m^*(x) - m(x, p(x, y))| \leqslant c \cdot |(m^*(z) - m(z, y))|.$$

- in case $r(x)$ has the form $1z$, then z is an optimal feasible solution of x.

In the situation of Definition 13, we say r maps x to instance z in case $r(x)$ is $0z$.

We fix some appropriate effective enumeration $(r_0, p_0, c_0), (r_1, p_1, c_1), (r_2, p_2, c_2), \ldots$ of all tuples of functions r_i and p_i in $\mathscr{F}\mathscr{P}$ and a positive rational c_i. Informally, we denote such a tuple (r, p, c) as $\leqslant_L$-reduction with reduction function r and pull-back function p, and in case P, Q, r, p and c satisfy Definition 13, we say that the fact $P \leqslant_L Q$ is witnessed by (r, p, c). By definition of $\leqslant_L$, a fact $P \leqslant_L Q$ holds iff it is witnessed by some $\leqslant_L$-reduction in our enumeration.

If restricted to optimization problems where the respective sets of instances are non-empty and do not contain degenerated instances, then the reducibility $\leqslant_L$ as defined in Definition 13 coincides with the original, more restrictive definition in [14]. Given such optimization problems P and Q and some witnessing $\leqslant_L$-reduction (r, p, c) which satisfies Definition 13, we can change r and p such that they still reduce P to Q, but satisfy in addition the more restrictive original definition of $\leqslant_L$. For instances x where $r(x)$ is equal to $1z$, that is, where z is an optimal solution for x, we change the reduction function r such that it maps x to some fixed instance of P, and we change the pull-back function such that for all y it maps (x, y) to the optimal solution z. By assumption every instance x of P has a non-empty set of solutions and thus $r(x)$ cannot be the empty string due to the definition of r.

The relation $\leqslant_L$ as defined in Definition 13 is still in accordance with the motivation for introducing reducibilities between optimization problems: if an optimization problem P is $\leqslant_L$-reducible to an optimization problem Q where for the latter we have a polynomial time approximation algorithm, then by combining this algorithm and a witnessing reduction we obtain a polynomial time approximation algorithm for P.

Khanna et al. [7] introduce a variant $\leqslant_E$ of the relation $\leqslant_L$ where in particular they require the respective optimal solutions of an instance and of its image under the reduction function to be related by a polynomial instead of a constant factor. Like in the case of the reducibility $\leqslant_L$, we consider an adjusted version of the relation $\leqslant_E$ which takes care of optimization problems which have no instances at all or have degenerated instances. While the subsequent embedding results hold for the relations $\leqslant_E$ and $\leqslant_L$ alike, we give proofs only for the case of the relation $\leqslant_L$, and omit the almost identical considerations for the relation $\leqslant_E$. In this connection, recall from Section 1.2 that our methods do not extend to relations such as *AP*-, *P*-, or *PTAS*-reducibility.

3.2. Embeddings of partial orderings into $\mathscr{N}\mathscr{P}\mathscr{O}$

In order to apply the results and techniques for bounded relations on ω^ω from Section 2.4 to reducibilities between optimization problems, we propose a representation of maximization problems by functions in ω^ω. For reasons to be explained below, we have to treat maximization and minimization problems separately. So we restrict our exposition to maximization problems, and note that the case of minimization problems can be handled similarly.

We define a mapping Ψ from ω^ω to maximization problems, where in particular the class $\mathscr{F}\mathscr{P}$ of functions computable in polynomial time is mapped onto the class of

$\mathcal{NP}$ maximization problems. We let the function s from ω^3 to ω be defined by

$$s(e,y,t) := \begin{cases} \Phi_e(y) & \text{if } |y| \leqslant |t| \text{ and } T_e(y) \text{ converges in less than } |t| \text{ steps,} \\ 0 & \text{otherwise.} \end{cases}$$

Here we denote by Φ_e the partial recursive function from ω to ω computed by the eth Turing machine T_e. Then for each function g in ω^ω, we obtain a maximization problem $\Psi(g)$ where for all x and for $g(x) = \langle i, e_0, e_1, t\rangle$

- the string x is an instance of $\Psi(g)$ iff i differs from 0,
- a string y is a feasible solution for x iff $s(e_0, y, t)$ differs from 0 and in this case we let

$$m(x,y) := 1 + s(e_1, y, t).$$

Here $\langle .,.,.,.\rangle$ is the usual bijection from ω^4 onto ω which is computable as well as invertible in polynomial time. The definition of the mapping Ψ can be adjusted in order to handle additional requirements in the definition of the concept optimization problem such as the set of feasible solutions for each instance being prefix-free. We leave to the reader the routine task to check that the function Ψ maps the class $\mathcal{FP}$ of functions computable in polynomial time onto the class of maximization problems in $\mathcal{NPO}$. The mapping Ψ and the reducibility $\leqslant_L$ together induce a partial preordering $\leqslant_L^\Psi$ on ω^ω defined by

$$f \leqslant_L^\Psi g \quad \text{iff} \quad \Psi(f) \leqslant_L \Psi(g). \tag{8}$$

Occasionally, we extend notation introduced in connection with the relation $\leqslant_L$ to the relation $\leqslant_L^\Psi$ and in particular we say a fact $f \leqslant_L^\Psi g$ is witnessed by some $\leqslant_L$-reduction if this reduction witnesses $\Psi(f) \leqslant_L \Psi(g)$.

The mapping Ψ yields canonically an isomorphism between the two degree structures which are induced, firstly, by $\leqslant_L^\Psi$ on $\mathcal{FP}$ and, secondly, by $\leqslant_L$ on the class of maximization problems in $\mathcal{NPO}$. We will show next that the relation $\leqslant_L^\Psi$ satisfies the assumption of Theorem 11, and that hence for both degree structures every countable partial ordering can be embedded into every proper interval. A similar remark holds for the relation $\leqslant_L^\Psi$ which is defined from the relation $\leqslant_E$ as in (8).

Corollary 14. *Let $\leqslant_r$ be equal to the relation $\leqslant_E$ or $\leqslant_L$. Then for the degree structure induced on $\mathcal{NPO}$ by the relation $\leqslant_r$ every countable partial ordering can be embedded into every proper interval between two maximization problems, and the same holds for proper intervals between minimization problems.*

In connection with the fact that we are able to show Corollary 14 only for intervals between either two maximization or two minimization problems, recall that we obtain intermediate sets as required by combining the optimization problems which bound the given interval according to a definition by cases w.r.t. some gap language. Now, if we were combining a maximization and a minimization problem, we would obtain an "optimization problem" where the goal varies with the instances.

Proof. We show, firstly, that the relation $\leqslant_L^\Psi$ satisfies the assumption of Theorem 11 and, secondly, that in case we apply Theorem 11 to a proper interval which is bounded by two functions in $\mathscr{FP}$, then also the constructed intermediate functions are contained in $\mathscr{FP}$. Corollary 14 then follows by definition of $\leqslant_L^\Psi$ because composing an embedding of some partial ordering into $\mathscr{FP}$ with the function Ψ yields an embedding of this partial ordering into the substructure of $\mathscr{NPO}$ formed by all maximization problems, and hence yields an embedding of the partial ordering into the structure formed by all of $\mathscr{NPO}$.

Recall that we have fixed an enumeration $(r_0, p_0, c_0), (r_1, p_1, c_1), (r_2, p_2, c_2), \ldots$ of $\leqslant_L$-reductions. From this enumeration, we obtain a recursive predicate R which witnesses that $\leqslant_L^\Psi$ is a bounded relation. Here $R(f, g, e, x)$ is true iff the functions r_e and p_e and the rational c_e witness that instance x of the maximization problem $\Psi(f)$ is reduced to $\Psi(g)$ in the way required by Definition 13.

Next, we show that the class $\mathscr{M}_L^\Psi$ contains the simulation class $\mathscr{P}$ and hence is a simulation class itself. Given functions f and f' in ω^ω such that $\Psi(f)$ and $\Psi(f')$ are both $\leqslant_L$-reducible to $\Psi(g)$ with g in ω^ω, we choose witnessing $\leqslant_L$-reductions (r', p', c') and (r'', p'', c'') in the above enumeration. Then given some set M which is computable in polynomial time, we have

$$\Psi(\langle f, f'\rangle^M) \leqslant_L \Psi(g)$$

via the $\leqslant_L$-reduction (r, p, c) where c is equal to the maximum of c' and c'' and where we let

$$r(x) := \begin{cases} r'(x) & \text{if } x \text{ is in } M,\\ r''(x) & \text{otherwise,}\end{cases} \qquad p(x, y) := \begin{cases} p'(x, y) & \text{if } x \text{ is in } M,\\ p''(x, y) & \text{otherwise.}\end{cases} \tag{9}$$

Then, as $\mathscr{M}_L^\Psi$ contains the simulation class $\mathscr{P}$, we can choose the sets R_i, which we have used in (5) while defining Π_0, to be in $\mathscr{P}$, which then results in the range of Π_0 being contained in $\mathscr{FP}$.

The relation $\leqslant_L^\Psi$ is c.f.v. Assume that f_1 and g_1 are finite variations of f_0 and g_0, respectively, where $f_0 \leqslant_L^\Psi g_0$ is witnessed by (r, p, c). Then we can change r and p in order to obtain a witness for $f_1 \leqslant_L^\Psi g_1$: firstly, for the finitely many places x where f_0 differs from f_1, the new pull-back function outputs an optimal solution for instance x of $\Psi(f_1)$ and, secondly, for each of the finitely many places w where g_0 differs from g_1 and for all x where $r(x)$ is equal to w, the new pull-back function on input (x, y) outputs $p(x, y_0)$ where y_0 is an optimal solution for instance w of $\Psi(g_0)$.

So it remains to show that $\leqslant_L^\Psi$ is a faithful partial preordering. Every constant function f is $\leqslant_L^\Psi$-reducible to all other functions as is witnessed by the polynomial time computable reduction function which returns always an optimal solution in case the identical instances of $\Psi(f)$ are non-degenerated or, otherwise, returns always the empty string. We leave it to the reader to show that the relation $\leqslant_L^\Psi$ is reflexive and transitive. Observe in this connection that transitivity holds because we can compose two $\leqslant_L$-reductions via composing the forward and pull-back functions, respectively, in the natural way except for some straightforward special actions which have to be taken in case one of the forward functions does not map its argument to an instance but yields instead an optimal solution or the empty string. Furthermore $\leqslant_L^{\Psi}$ has the join as a least upper bound operator as follows by a standard proof, using combinations of $\leqslant_L$-reductions similar to the ones introduced in (9). $\square$

3.3. Lattice embeddings into $\mathscr{NPO}$

We show that for the relations $\leqslant_L$ and $\leqslant_E$ the result on embeddings of partial orderings stated as Corollary 14 extends to embeddings of countable distributive lattices.

Theorem 15. *Let $\leqslant_r$ be equal to the relation $\leqslant_L$ or $\leqslant_E$. Then for the degree structure induced on $\mathscr{NPO}$ by the relation $\leqslant_r$ every countable distributive lattice can be embedded with least or greatest element preserved into every proper interval between either two maximization or two minimization problems.*

Note that Theorem 15 extends by the same proof from embeddings into $\mathscr{NPO}$ to lattice embeddings into arbitrary proper intervals of the structure of recursive optimization problems. As in the setting of polynomial time bounded reducibilities considered in [1], Theorem 15 yields several corollaries. In case $\mathscr{NPO}$ does not collapse to a single degree, we obtain, besides the existence of minimal pairs stated in Corollary 16, the existence of countable chains and anti-chains. Further, every degree which is not above all other degrees is meet-reducible, that is, is the greatest lower bound of two other degrees, and likewise, every degree which is not below all other degrees is join-reducible, that is, is the least upper bound of two other degrees.

In connection with Corollary 16 observe that there is a non-empty class of least optimization problems w.r.t. the relation $\leqslant_L$, that is, there is a least $\leqslant_L$-degree $\mathscr{L}_L$. In this situation, two optimization problems are denoted as minimal pair iff neither of them is contained in $\mathscr{L}_L$, but every optimization problem which is $\leqslant_L$-reducible to both of them has to be in $\mathscr{L}_L$. Corollary 16 is immediate from Theorem 15 by embedding the four element lattice into the interval between the given optimization problem and some least optimization problem (which has the same goal) with least element preserved.

Corollary 16. *Every $\mathscr{NP}$ optimization problem which is not $\leqslant_L$-reducible to all other optimization problems bounds a minimal pair of $\mathscr{NP}$ optimization problems.*

The proof of Theorem 15 is basically the same as the one which has been used by Ambo-Spies [1] for showing a corresponding result about polynomial time bounded reducibilities, and which has been applied with an axiomatic approach to resource bounded reducibilities in [12, 13]. While we give the proof in its entirety, we state in greater detail considerations which are related to specific features of the reducibilities considered here, and give only a brief account of the remaining parts.

Proof of Theorem 15. We consider the relation $\leqslant_L$ and leave the almost identical considerations for $\leqslant_E$ to the interested reader. We show first that the statement of the theorem holds with $\mathscr{NPO}$ and $\leqslant_L$ replaced by $\mathscr{FP}$ and $\leqslant_L^\Psi$, respectively. We then argue that by construction the corresponding lattice embeddings into $\mathscr{FP}$ yield lattice embeddings into $\mathscr{NPO}$ if composed with Ψ.

It is known from lattice theory that every countable distributive lattice can be embedded as a lattice into the countable atomless Boolean algebra with least and greatest element preserved, see [1] and the references given there. Now lattice embeddings compose, and thus we are done if we can embed the countable atomless Boolean algebra in the required way. We first construct such an embedding which preserves the least element, and then indicate the minor changes necessary in case we want to preserve the greatest element. We let

$$3 \cdot \mathscr{P} := \{3 \cdot X : X \text{ in } \mathscr{P}\} \quad \text{where } 3 \cdot X := \{3 \cdot x : x \text{ in } X\}.$$

The class $3 \cdot \mathscr{P}$ contains exactly the sets computable in polynomial time which contain only multiples of three. A similar argument as for the structure $(\mathscr{P}^*, \leqslant^*)$ shows that

$$((3 \cdot \mathscr{P})^*, \leqslant^*) := (\{[X]) : X \text{ in } 3 \cdot \mathscr{P}\}, \leqslant^*)$$

is the countable atomless Boolean algebra. The structure $((3 \cdot \mathscr{P})^*, \leqslant^*)$ is a sublattice of $(\mathscr{P}^*, \leqslant^*)$ (however, it is not a subalgebra because the greatest elements in both structures are different). Thus the mapping Π constructed in the proof of Theorem 11 is not only a order embedding of $(\mathscr{P}^*, \leqslant^*)$, but also of $((3 \cdot \mathscr{P})^*, \leqslant^*)$. So we are done if we are able to show, firstly, that the embedding Π respects least upper bounds and, secondly, that we can arrange in the case of $\leqslant_L^\Psi$ that the restriction of Π to $(3 \cdot \mathscr{P})^*$ respects also greatest lower bounds.

Claim 1. *The function Π respects least upper bounds.*

Proof. Given sets A_i and A_j in $\mathscr{P}$, the l.u.b. of $[A_i]$ and $[A_j]$ in $(\mathscr{P}^*, \leqslant^*)$ is $[A_i \cup A_j]$. Now, the mapping Π_0 is a partial order embedding, and thus the function $\Pi_0(A_i \cup A_j)$ is an upper bound for $\Pi_0(A_i)$ and $\Pi_0(A_j)$. It remains to show that if the two latter functions are both reducible to some function g, then so is $\Pi_0(A_i \cup A_j)$. But this follows from

$$\begin{aligned} \Pi_0(A_i \cup A_j) &= \langle f \oplus g, f \oplus \emptyset \rangle^{R_i \cup R_j} = \langle f \oplus g, \langle f \oplus g, f \oplus \emptyset \rangle^{R_j} \rangle^{R_i} \\ &= \langle f \oplus g, \Pi_0(A_j) \rangle^{R_i} = \langle \Pi_0(A_i), \Pi_0(A_j) \rangle^{R_i} \leqslant_L^\Psi g, \end{aligned}$$

where the relations hold by definition of Π_0 and the choice of the sets R_k, by the properties of the function $\langle .,. \rangle$, by definition of $\Pi_0(A_j)$, because $\Pi_0(A_i)$ agrees with $f \oplus g$ on all numbers in R_i, and finally by assumption on g and because R_i is in $\mathscr{M}_L^\Psi$.

In the remainder of this proof, we say that every $\leqslant_L$-reduction to a function g is witnessed by a reduction function r which satisfies certain conditions iff for every fact

$f \leqslant_L^{\Psi} g$ there is some witnessing tuple (r, p, c) from our enumeration of $\leqslant_L$-reductions where r satisfies the conditions under consideration. Recall from the remark following Definition 13 that we say a reduction function r maps instance x to instance z in case $r(x)$ is equal to $0z$.

Claim 2. *For every recursive function ϕ_e in ω^ω, there are non-decreasing and unbounded recursive functions b_e and d from ω to ω such that every $\leqslant_L^{\Psi}$-reduction to ϕ_e is witnessed by a reduction function r such that firstly, for all x, we have $b_e(x) \leqslant x < d(x)$ and, secondly, for almost all x, in case x is mapped by r to instance z we have*

$$b_e(x) \leqslant z < d(x).$$

Here an index for b_e can be obtained effectively from e whenever ϕ_e is total.

Proof. We obtain a suitable upper bound d by letting $d(0)$ be equal to 1, and by

$$d(x+1) := 1 + \max[\{z\colon\ r_i(y) = 0z \text{ for some } i, y \leqslant x\} \cup \{x, d(x)\}],$$

where the functions $r_0, \ldots, r_n$ are the reduction functions from our enumeration of all $\leqslant_L$-reductions.

Given e and x in ω, we let $b_e(x)$ be the maximal number strictly less than x such that a total of $|x|$ steps is sufficient to compute by means of a brute-force algorithm for all z less than or equal to $b_e(x)$ an optimal solution y_0 for instance z of $\Psi(\phi_e)$. For total ϕ_e, the function b_e can be computed from e, and is by definition non-decreasing and unbounded. Then, assuming $P \leqslant_L [\Psi(\phi_e)]$ for some optimization problem P we choose some witnessing reduction (r, p, c) from our enumeration. By changing the reduction function r such that for all places x where r maps x to instance z with $z < b_e(x)$, the new reduction function first computes an optimal solution y_0 for z and then outputs an optimal solution $p(x, y_0)$ for instance x, we obtain a reduction (r', p, c) in our enumeration which again witnesses $P \leqslant_L [\Psi(\phi_e)]$ and which satisfies the requirements from the claim.

Given some gap language G and a place x, we denote by $Nb(x, G)$ the union of the three consecutive blocks of G such that x is contained in the middle block.

Claim 3. *For every recursively presentable subclass $\mathscr{C}$ of ω^ω there is a recursive gap language G_0 such that every $\leqslant_L^{\Psi}$-reduction to some function g in $\mathscr{C}$ is witnessed by a reduction function r such that for all x in ω, in case x is mapped to instance z, then z is in $Nb(x, G)$.*

Proof. We fix some appropriate effective enumeration $e_0, e_1, \ldots$ such that the recursively presentable class $\mathscr{C}$ is equal to $\{\phi_{e_0}, \phi_{e_1}, \ldots\}$. We construct the gap language G_0 in stages. During stage s we specify block s of G_0. This then determines G_0 by letting $G_0(0)$ be equal to 0. At stage 0 we let block 0 of G_0 be equal to $\{0\}$. At stage

$s>0$ we specify block s by determining its maximal element w_s where, firstly, w_s is strictly larger than w_{s-1}, secondly, $d(w_{s-1})$ is less than w_s, and, thirdly, for every index e in $\{e_1,\ldots,e_s\}$ the value $b_e(w_s)$ is contained in block s. Here we choose the functions b_e and d according to Claim 2. We leave it to the reader to verify that G_0 has the required properties. Here witnessing reductions can be obtained from Claim 2. However, given an $\leqslant_L$-reduction which witnesses a reduction to some function ϕ_e according to Claim 2, then in order to obtain a witness for Claim 3, for finitely many places we have to code optimal solutions into the reduction function because for every given e, firstly, according to Claim 2 finitely many x might be mapped to instances outside the bounds given by b_e and d and, secondly, b_e is not considered while defining blocks $s<e$ of G_0.

We know from the proof of Corollary 14 that the class $\mathscr{M}_L^{\Psi}$ contains the simulation class $\mathscr{P}$. Thus we can assume that the sets R_i, which we have obtained from the coding lemma and which we have used while defining the embedding Π_0, are all in $\mathscr{P}$ and that, consequently, the range of Π_0 is contained in the recursively presentable class

$$\mathscr{C} := \{\langle f \oplus g, f \oplus \emptyset\rangle^R\colon R \text{ in } \mathscr{P}\}.$$

We apply Claim 3 to the class $\mathscr{C}$ and obtain a recursive gap language G_0. The statement of Claim 3 remains valid if we replace G_0 with some gap cover G of G_0, because then for every x the set $Nb(x,G_0)$ is contained in $Nb(x,G)$. Further, all properties of the embeddings Π_0 and Π shown so far remain valid if we apply the definition of Π_0 not directly to the gap language G_1 obtained from the diagonalization lemma, but to some recursive gap language G which is simultaneously a gap cover for G_0 and for G_1. So we are done, if we show that this change in the definition of Π_0 entails Claim 4.

Claim 4. *The restriction of the function Π to the class $(3\cdot\mathscr{P})^*$ respects greatest lower bounds.*

Proof. Given sets A_i and A_j which are finite variations of sets in $3\cdot\mathscr{P}$, the greatest lower bound of $[A_i]$ and $[A_j]$ in $(3\cdot\mathscr{P}^*, \leqslant^*)$ is $[A_i\cap A_j]$. The mapping Π_0 is a partial order embedding, and thus the set $\Pi_0(A_i\cap A_j)$ is a lower bound for $\Pi_0(A_i)$ and for $\Pi_0(A_j)$. It remains to show that if some function f is reducible to both of the two latter functions, then it is also reducible to $\Pi_0(A_i\cap A_j)$. The images of A_i and A_j under Π_0 are in the class $\mathscr{C}$, and thus we can choose reductions (r',p',c') and (r'',p'',c'') which witness the reductions from f to A_i and A_j, respectively, and where instances z selected by the reduction functions via $r(x)=0z$ and $r'(x)=0z$ are always contained in $Nb(x,G_0)$, and thus are also contained in $Nb(x,G)$ and in $Nb(x,M)$, where M is the gap cover for G obtained from the coding lemma. Now, for almost all n, the sets A_i and A_j do not contain the numbers $3n-1$ and $3n+1$. As a consequence for almost all blocks s of M where for some n in ω, the number s is in the set $S := \{3n-1, 3n, 3n+1\}$ we find

- in case $3n$ is in A_j, the set $A_i\cap A_j$ agrees on S with A_i, and consequently $\Pi_0(A_i\cap A_j)$ agrees on blocks $3n-1$, $3n$, and $3n+1$ of M with $\Pi_0(A_i)$,

- in case $3n$ is not in A_j, the set $A_i \cap A_j$ agrees on S with A_j, and consequently $\Pi_0(A_i \cap A_j)$ agrees on blocks $3n-1$, $3n$, and $3n+1$ of M with $\Pi_0(A_j)$.

We choose A_k in $\mathscr{P}$ where

$$A_k := \{3n-1, 3n, 3n+1 \colon 3n \text{ is in } A_j\} \setminus \{-1\}.$$

For the corresponding set R_k obtained from the coding lemma, we have that for almost all x in R_k, the function $\Pi_0(A_i \cap A_j)$ agrees on $Nb(x, M)$ with $\Pi_0(A_i)$, and for almost all x not in R_k, the function $\Pi_0(A_i \cap A_j)$ agrees on $Nb(x, M)$ with $\Pi_0(A_j)$. By assumption on the chosen witnessing reductions, we thus obtain an $\leqslant_L$-reduction (r, p, c) which witnesses that f is reducible to a finite variation of $\Pi_0(A_i \cap A_j)$ by letting c be equal to the maximum of c' and c'' and by

$$r(x) := \begin{cases} r'(x) & \text{if } x \text{ is in } R_k, \\ r''(x) & \text{otherwise,} \end{cases} \qquad p(x, y) := \begin{cases} p'(x, y) & \text{if } x \text{ is in } R_k, \\ p''(x, y) & \text{otherwise.} \end{cases}$$

Now let $\tilde{\Psi}$ be the composition of Π and Ψ. Like in the proof of Theorem 11 we infer that $\tilde{\Psi}$ embeds $((3 \cdot \mathscr{P})^*, \leqslant^*)$ as a partial ordering into $(\mathscr{NPO}, \leqslant_L)$. Further, by construction of Π, the mapping $\tilde{\Psi}$ is a lattice embedding of $((3 \cdot \mathscr{P})^*, \leqslant^*)$ into the substructure of $(\mathscr{NPO}, \leqslant_L)$ induced by all maximization problems in $\mathscr{NPO}$. It remains to show that $\tilde{\Psi}$, in fact, is a lattice embedding into the full structure which includes also the minimization problems in $\mathscr{NPO}$, that is, the mapping $\tilde{\Psi}$ respects least upper and greatest lower bounds not only w.r.t. maximization problems, but also w.r.t. minimization problems in $\mathscr{NPO}$. But this follows by arguments similar to the case of maximization problems. First, in case two problems $\tilde{\Psi}([A_i])$ and $\tilde{\Psi}([A_j])$ are both reducible to a minimization problem P, then so is $\tilde{\Psi}([A_i \cup A_j])$. Here a witnessing reduction is obtained as in the proof of Claim 1 by alternating between the two assumed reductions to P. Second, in case a minimization problem P is reducible to both of $\tilde{\Psi}([A_i])$ and $\tilde{\Psi}([A_j])$, then P is also reducible to $\tilde{\Psi}([A_i \cap A_j])$. Here again a witnessing reduction is obtained as in the proof of Claim 4 by alternating between the two assumed reductions while taking into account that we can arrange that both reductions map an instance x only to instances in $Nb(x, M)$.

The mapping Π_0 takes the empty set to $f \oplus \emptyset$ and thus preserves the least element of the embedded structure $(3 \cdot \mathscr{P}^*, \leqslant^*)$. In case we want to preserve the greatest element, we embed instead the structure $(\mathscr{Q}^*, \leqslant^*)$ where

$$\mathscr{Q} := \{X \cap \omega \colon X \text{ is in } 3 \cdot \mathscr{P}\},$$

that is, intuitively speaking, we construct an embedding Π_0 where the non-coding gaps its images are equal to $f \oplus g$ instead of $f \oplus \emptyset$. $\square$

4. Decidability

In view of the representation of optimization problems in $\mathscr{NPO}$ by functions in $\mathscr{FP}$ via the mapping Ψ, one might ask whether given an index (w.r.t. the standard enumeration of $\mathscr{FP}$) for some function f in $\mathscr{FP}$, it is possible to decide effectively on properties of the optimization problem $\Psi(f)$ such as being approximable in polynomial time within a constant factor. By a straightforward adaptation of work by Schmidt [15], we will show below that this question has to be answered negatively. Before we state this result we extend the standard concept of recursively presentable classes of sets to subclasses of ω^ω and to classes of maximization problems. Here the restriction to maximization problems again relates to the fact that the we have chosen the embedding Ψ such that its image contains only maximization problems. We assume that the results of this section carry over to classes of minimization problems and, at the cost of slightly more technical formulations, can even be adapted to mixed classes of maximization and minimization problems.

Definition 17.
- Given a function g in ω^ω, let for all i in ω the function $g^{[i]}$ be defined by $g^{[i]}(x) := g(\langle i, x\rangle)$.
- A subclass of $\mathscr{FP}$ is RECURSIVELY PRESENTABLE iff it either is empty or is equal to $\{g^{[i]}\colon i \text{ in } \omega\}$ for some recursive function g in ω^ω.
- A class of maximization problems is RECURSIVELY PRESENTABLE iff it is equal to $\{\Psi(f)\colon f \text{ in } \mathscr{C}\}$ for some recursively presentable subclass $\mathscr{C}$ of ω^ω.
- A class $\mathscr{C}$ of maximization problems is C.F.V. iff the class $\{f\colon \Psi(f) \text{ in } \mathscr{C}\}$ is c.f.v.

Proposition 18. *Let $\mathscr{D}_0$ and $\mathscr{D}_1$ be proper subclasses of the class of maximization problems in $\mathscr{NPO}$ such that both classes are c.f.v. and their union contains all maximization problems in $\mathscr{NPO}$. Then one of the classes $\mathscr{D}_0$ and $\mathscr{D}_1$ is not recursively presentable.*

Proof. Assume for a proof by contradiction that $\mathscr{D}_0$ and $\mathscr{D}_1$ were recursively presentable, and that these facts are witnesses by recursively presentable subclasses $\mathscr{C}_0$ and $\mathscr{C}_1$ of $\mathscr{FP}$. We choose functions h_0 in $\mathscr{C}_0$ and h_1 in $\mathscr{C}_1$ such that $\Psi(h_0)$ is not in $\mathscr{D}_1$ and $\Psi(h_1)$ is not in $\mathscr{D}_0$. By a standard diagonalization construction which is similar to the one used in the proof of Lemma 4 and which in particular exploits the fact that $\mathscr{D}_0$ and $\mathscr{D}_1$ are c.f.v., we construct a recursive gap language G such that for all i, firstly, $\Psi(h_0)$ differs from the Ψ-image of each of the first i functions in $\mathscr{C}_1$ at some x in block i of G, and secondly, $\Psi(h_1)$ differs from the Ψ-image of each of the first i functions in $\mathscr{C}_0$ at some x in block i of G. As in the coding theorem we choose a gap cover M for G which is computable in polynomial time and we let h be equal to $\langle h_0, h_1\rangle^M$. Now we obtain a contradiction, because h is in $\mathscr{FP}$, but by construction $\Psi(h)$ differs from all maximization problems in $\mathscr{D}_0$ and $\mathscr{D}_1$. $\square$

The proof of Proposition 18 is similar to the proof of a corresponding result about recursively presentable classes of sets due to Schmidt [15]. In connection with Corollaries 19 and 20 recall that a maximization problem can be approximated (in polynomial time) within a constant factor iff there is a rational $\delta > 0$ and a polynomial time bounded algorithm which given as input a non-degenerated instance x outputs a feasible solution y for x such that the quality $m(y)$ is at least δ times the quality of an optimal solution.

Corollary 19. *Let $\mathscr{D}_{\mathrm{app}}$ and $\mathscr{D}_{\mathrm{non}}$ be the classes of maximization problems in $\mathscr{NPO}$ which can and which cannot, respectively, be approximated within a constant factor. Then $\mathscr{D}_{\mathrm{app}}$ is recursively presentable, while $\mathscr{D}_{\mathrm{non}}$, if not empty, is not.*

Corollary 20. *Let $f_0, f_1, \ldots$ be the standard enumeration of $\mathscr{FP}$ and assume that there are maximization problems in $\mathscr{NPO}$ which cannot be approximated within a constant factor. Then there is no effective procedure which decides for a given index i in ω whether the maximization problem $\Psi(f_i)$ is approximable within a constant factor.*

In order to show Corollary 20, assume for a contradiction that there were a procedure as required in the lemma and let h be a function in $\mathscr{FP}$ such that $\Psi(h)$ is not approximable within a constant factor. Then we can easily define a recursive function g where its ith row $g^{[i]}$ is equal to f_i in case f_i is not approximable within a constant factor, and is equal to h, otherwise. But then g witnesses that $\mathscr{D}_{\mathrm{non}}$ is recursively presentable, thus contradicting Corollary 19.

The construction of the function g in the proof of Corollary 19 is an adaptation of standard techniques from complexity theory which are used in connection with recursively presentable classes of sets.

Proof of Corollary 19. In case $\mathscr{D}_{\mathrm{non}}$ is non-empty, the classes $\mathscr{D}_{\mathrm{app}}$ and $\mathscr{D}_{\mathrm{non}}$ satisfy the assumption of Proposition 18. Hence in order to show the corollary it suffices to show that $\mathscr{D}_{\mathrm{app}}$ is recursively presentable, that is, it suffices to construct a recursive function g such that $\mathscr{D}_{\mathrm{app}}$ is equal to $\{\Psi(g^{[i]})\colon i \in \omega\}$. So let $f_0, f_1, \ldots$ be the effective standard enumeration of the functions in $\mathscr{FP}$, and given $i = \langle k, l, m\rangle$ define $g^{[i]}$ by

$$g^{[i]}(x) = \begin{cases} f_k(x) & \text{in case } f_l \text{ approximates } \Psi(f_k) \\ & \quad \text{within a factor of } 1/m \text{ for all } z \leqslant |x|, \\ \mathrm{o}(x) & \text{otherwise.} \end{cases}$$

Here $\Psi(o)$ is some easily approximable problem, say, where for all x and y of equal length, y is a feasible solution for the instance x with quality 1. Now, first, for all i in ω, for $i = \langle k, l, m\rangle$ we find that $g^{[i]}$ is a finite variation of f_k or of o. Thus in particular $\Psi(g^{[i]})$ is indeed an $\mathscr{NP}$ maximization problem. Moreover, this maximization problem can be approximated within a constant factor because in case $g^{[i]}$ is finite variation of o there is a trivial approximation and, otherwise, we infer from the definition of g that f_l must witness that $\Psi(g^{[i]})$ is approximable within a factor of $1/m$. Second, given a maximization problem Q in $\mathscr{D}_{\mathrm{app}}$, choose k, m, and l such that Q is equal to $\Psi(f_k)$ and f_l approximates Q within a factor of $1/m$. Then by construction of g, we have $Q = \Psi(g^{[i]})$ with $i = \langle k, l, m\rangle$. $\square$

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